Lecture 5 on Sept. 19

In the last lecture, we have studied the fundamental theorem of algebra. Now we study the second theorem associated with polynomials, the Lucas' theorem. Before that let us consider an efficient way to calculate derivatives for a holomorphic function $f(z)$. From the last lecture, particularly by the change of variables in (0.2) of the last lecture note, we know that if f is holomorphic in Ω, then $\partial_{\bar{z}} f \equiv 0$ in Ω. Now we calculate $\partial_z f$. In fact by chain rule, we have

$$\partial_z = \partial_x \frac{\partial x}{\partial z} + \partial_y \frac{\partial y}{\partial z} = \frac{1}{2}\,\partial_x + \frac{1}{2i}\,\partial_y.$$

Therefore it holds

$$\partial_z f = \frac{1}{2}\partial_x(u+iv) + \frac{1}{2i}\partial_y(u+iv) = \frac{1}{2}(\partial_x u + \partial_y v) + \frac{i}{2}(\partial_x v - \partial_y u).$$

By (0.1) in the last lecture note, we have

$$\partial_z f = \partial_x u + i\,\partial_x v.$$

By the calculation at the beginning of the last lecture note, we show that

$$\partial_z f = f'(z).$$

the above equality tells us that we can treat the derivative of a holomorphic function $f(z)$ as the derivatives for single variable functions.

Example 1: If $P_n = a_0 + a_1 z + ... + a_n z^n$, then $P_n'(z) = a_1 + ... + a_n n z^{n-1}$.

Example 2: If P_n is given by

$$P_n(z) = c\,\Pi_{k=1}^{n}(z-\alpha_k),$$

then by product rule and the above arguments, we have

$$P_n'(z) = c\sum_{k=1}^{n}(z-\alpha_1)\cdot ....\widehat{(z-\alpha_k)}\cdot ...(z-\alpha_n).$$

Here $(z-\alpha_1)\cdot ....\widehat{(z-\alpha_k)}\cdot ...(z-\alpha_n)$ is the product without the term $z-\alpha_k$.

By Example 2 above, we get

$$\frac{P_n'(z)}{P_n(z)} = \sum_{k=1}^{n}\frac{1}{z-\alpha_k}.$$

Now we assume all roots of P_n satisfy

$$\mathrm{Im}\left(\frac{\alpha_k - a}{b}\right) > 0, \qquad \text{for all } k = 1, ..., n.$$

If z^* is an arbitrary point such that

$$\mathrm{Im}\left(\frac{z^* - a}{b}\right) < 0,$$

then obviously

$$\mathrm{Im}\left(\frac{z^* - \alpha_k}{b}\right) < 0, \qquad \text{for all } k = 1, ..., n.$$

therefore

$$\text{Im}\left(\frac{bP_n'(z^*)}{P_n(z^*)}\right) = \sum_{k=1}^{n} \text{Im}\left(\frac{b}{z^* - \alpha_k}\right) > 0.$$

In other words, $P_n'(z^*) \neq 0$. The above arguments essentially imply that

Theorem 0.1 (Lucas theorem)**.** *all roots of P_n' must stay within the convex hull where all roots of $P_n(z)$ are contained.*

More detailed arguments of this theorem can be found from the textbook.

finally we consider the map $f(z) = z^n$. Given a disk with radius R, we separate it into n sections with equal angle $2\pi/n$. One can show that each section is mapped to a whole disk in $\mathbb{C}$ with radius R^n. If we circulate around the original point once then the image of the map $f(z) = z^n$ will circulate around the original point n times. Usually n is also called winding number in topology theory. Moreover one can show that for any $\alpha_0 \in \mathbb{R}$, z^n is an one-one correspondence between the region $\{\alpha_0 \leq \theta < \alpha_0 + 2\pi/n\}$ and $\mathbb{C}$. In the next lecture, we are going to study rational functions.